Data-Driven Identification of Stable Non-Linear Systems Using Long Short-Term Memory[1]

Research Seminar
Computer & Systems Engineering (M.Sc.)

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1. E. Terzi, F. Bonassi, M. Farina, and R. Scattolini, “Learning model predictive control with long short-term memory networks,” International Journal of Robust and Nonlinear Control, Apr. 2021, doi: https://doi.org/10.1002/rnc.5519. ↩

Motivation

Given is a **non-linear**, **time-invariant** system Σ with

$$\tilde{x}^+ = f(\tilde{x}, u)$$
$$\tilde{y} = h(\tilde{x})$$

which is **stable** and where $u \in [u_{min}, u_{max}]$.

Find a model for it using **only data of input** u **and output** $\tilde{y}$.

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Comparison functions[1]:

$$\begin{aligned}
\mathcal{K} \quad &:= \{\alpha : \mathbb{R}^+ \to \mathbb{R}^+ \quad &&|\, \alpha \text{ cont., str. inc., } \alpha(0) = 0\} \\
\mathcal{K}_\infty \quad &:= \{\alpha : \mathbb{R}^+ \to \mathbb{R}^+ \quad &&|\, \alpha \in \mathcal{K},\ \alpha \text{ unbounded}\} \\
\mathcal{KL} \quad &:= \{\beta : \mathbb{R}^+ \times \mathbb{N}_0 \to \mathbb{R}^+ \quad &&|\, \beta \text{ cont., } \beta(\cdot, k) \in \mathcal{K}, \\
& && \quad \beta(s, \cdot) \text{ str. dec., } \lim_{t\to\infty} \beta(s, k) = 0\}
\end{aligned}$$

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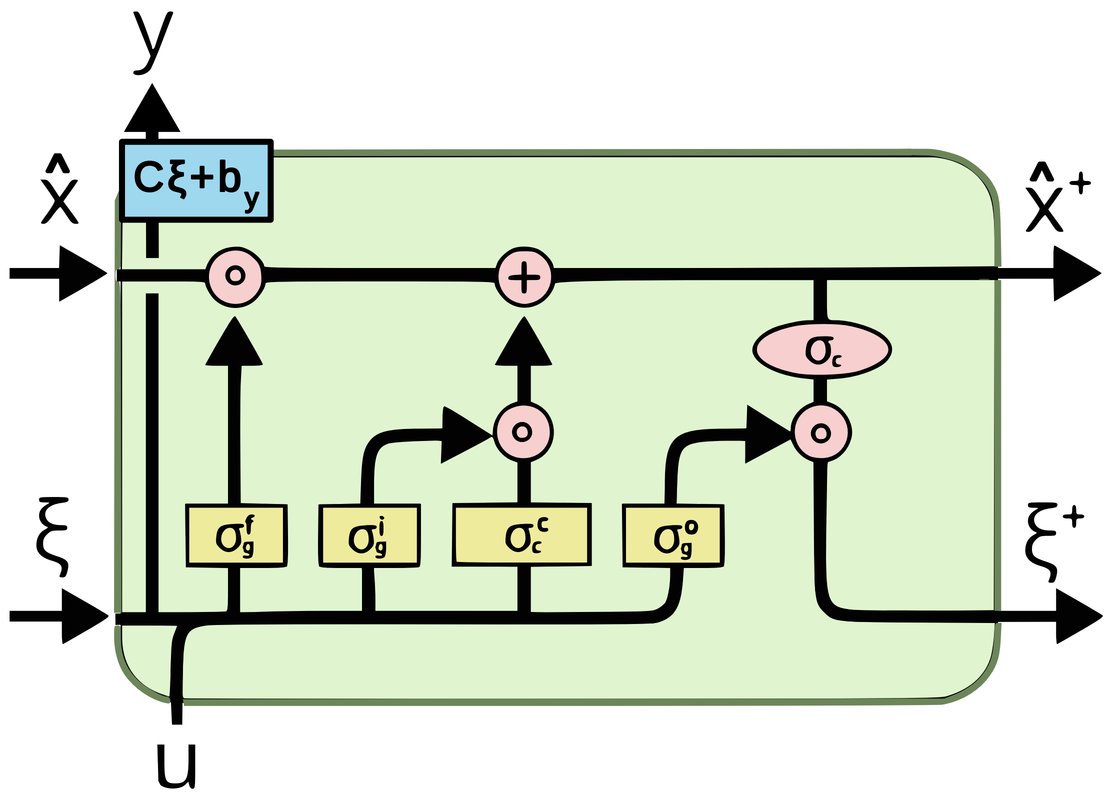
1. Lars Grüne and Jürgen Pannek, Nonlinear model predictive control : theory and algorithms. Switzerland: Springer, 2017. ↩

Long Short-Term Memory (LSTM)[1][2]



1. H. Sak, A. Senior, and F. Beaufays, "Long short-term memory recurrent neural network architectures for large scale acoustic modeling," Interspeech 2014, Sep. 2014, doi: https://doi.org/10.21437/interspeech.2014-80. ↩

2. C. Olah, "Understanding LSTM Networks." Colah's Blog. https://colah.github.io/posts/2015-08-Understanding-LSTMs/ (accessed Feb. 02, 2025). ↩

Stability of LSTM

We require the LSTM to be ISS/δISS. Reasons:

1. Σ is ISS/δISS.
2. Eventually,
   - (ISS) $x(k)$ gets near 0.
   - (δISS) $x_1(k), x_2(k)$ get near to each other.
3. Safety guarantees.

$\rightarrow$ **How to guarantee ISS/δISS for LSTM formally?**

Stability of LSTM

Theorem 1. *The LSTM network is ISS with respect to the input u and bias b_c if A is Schur, where*

$$A = \begin{bmatrix} \overline{\sigma}_g^f & \overline{\sigma}_g^i ||U_c|| \\ \overline{\sigma}_g^o \overline{\sigma}_g^f & \overline{\sigma}_g^o \overline{\sigma}_g^i ||U_c|| \end{bmatrix}.$$

Proposition 1. *A is Schur if and only if the following inequation holds:*

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Lemma 1. *Given a 2×2 real matrix A, it is Schur if and only if*

$$-1 - a < b < 1,$$

where $a = -A_{11} - A_{22}$ and $b = A_{11}A_{22} - A_{12}A_{21}$.

Stability of LSTM

Theorem 2. *The LSTM network is δISS with respect to the inputs u_1 and u_2 if A_δ is Schur, where*

$$A_\delta = \begin{bmatrix} \overline{\sigma}_g^f & \alpha \\ \overline{\sigma}_g^o \overline{\sigma}_g^f & \alpha \overline{\sigma}_g^o + \frac{1}{4} \overline{\sigma}_c^x ||U_o|| \end{bmatrix}$$

with

$$\alpha = \frac{1}{4} ||U_f|| \frac{\overline{\sigma}_g^i \overline{\sigma}_c^c}{1 - \overline{\sigma}_g^f} + \overline{\sigma}_g^i ||U_c|| + \frac{1}{4} ||U_i|| \overline{\sigma}_c^c.$$

Proposition 2. *A_δ is Schur if the following inequation hold:*

$$-1 + \overline{\sigma}_g^f + \alpha \overline{\sigma}_g^o + \frac{1}{4} \overline{\sigma}_c^x ||U_o|| \; < \; \frac{1}{4} \overline{\sigma}_g^f \overline{\sigma}_c^x ||U_o|| \; < \; 1.$$

How to Find Parameters Ensuring ISS?

Formulate the **equations from Prop. 1** as $r < 0$.

Extend the loss function to **force residual** $r < 0$[1].

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1. The shown loss function aims to fullfilling the inequations from Proposition 1 to enforce just ISS. For enforcing δISS, one needs to consider the inequations from Proposition 2, instead. The procedure, however, is the same. ↩

Conclusion

- An **LSTM** network can be used to **model non-linear systems**.
- For this, **ISS/δISS is a desirable property** for the LSTM network.
- ISS and δISS of an LSTM network **can be shown formally**.
- ISS and δISS **can be enforced during optimization**.